Chapter 23: Advanced algebra

Starter 23 [page 446]
By counting, the numbers of squares/rectangles are 9, 36, 30. Thus \( k = 4 \).

To prove the formula, select one corner of a square/rectangle at random. On an \( m \) by \( n \) grid, there are \( m + 1 \) possible choices for the \( x \) coordinate and \( n + 1 \) for the \( y \) coordinate, giving \( (m + 1)(n + 1) \) possibilities altogether.

Now choose a second corner, not in the same row or column as before; this can be done in \( mn \) ways. Thus there would seem to be \( m(m + 1)n(n + 1) \) choices altogether.

However, each different square/rectangle gets counted four times in this way. Thus the number is \( (m + 1)(n + 1) \div 4 \) and the result is proved.

Exercise 23.1 [page 450]
1 \( 3\sqrt{2} \) 2 \( 4\sqrt{2} \) 3 \( 5\sqrt{2} \)
4 \( 3\sqrt{5} \) 5 \( 5\sqrt{6} \) 6 \( 2\sqrt{6} \)
7 \( 3\sqrt{11} \) 8 \( 6\sqrt{3} \) 9 \( 6\sqrt{3} \)
10 \( 7\sqrt{2} \) 11 \( 3\sqrt{5} \) 12 \( 7\sqrt{2} \)
13 \( 4\sqrt{11} \) 14 \( 4\sqrt{2} \) 15 \( 12 + 4\sqrt{5} \)
16 \( 8 + 7\sqrt{2} \) 17 \( 22 \) 18 \( 22 + 11\sqrt{5} \)
19 \( \frac{2 + 3\sqrt{5}}{5} \) 20 \( 2 - \sqrt{7} \) 21 a) \(-1\)
b) \(-3(1 - \sqrt{2}) \)
23 \(-3(2 + \sqrt{5}) \) 24 a) \( 8 + 3\sqrt{5} \) b) \( 10 - 4\sqrt{3} \)
25 a) \( 4 + 2\sqrt{7} \) b) \( 18 + 2\sqrt{7} \) c) \( 6 + 6\sqrt{7} \)
26 \((12\sqrt{7} - 16)\) cm 27 \(-2 + \sqrt{11} \)
28 \( \frac{1 + \sqrt{5}}{2} \) 29 \( -3 \pm \sqrt{13} \) 30 \( -4 \pm \sqrt{10} \)
31 \( \frac{5\sqrt{17}}{2} \) 32 \( x = -3 - \sqrt{11} \) or \( x = \sqrt{11} - 3 \)

Exercise 23.2 [page 453]
1 \( 8x + 3 \) 2 \( 5x + 2 \) 3 \( 5x + 2 \)
4 \( 5x + 2 \) 5 \( 13x + 4 \) 6 \( 11x + 2 \)
7 \( 5x + 5 \) 8 \( 5x + 7 \) 9 \( 5x + 7 \)
10 \( (x + 1)(x + 2) \) 11 \( 3(x + 1)(x + 2) \) 12 \( 3(x + 1)(x + 2) \)
13 \( 3 \) 14 \( 2 \) 15 \( 3 \)
16 \( -2 \) 17 \( 3 \) 18 \( 5 \)
19 \( 4, \frac{3}{2} \) 20 \( 2, -\frac{4}{11} \)

Exercise 23.3 [page 455]
1 \( x + 3 \) 2 \( 3x + 5 \)
3 \( 2x + 1 \) 4 \( 2x + 1 \)
5 \( x + 2 \) 6 \( 5(x + 3)^2 \)
7 \( x + 10 \) 8 \( 3x + 2 \)
9 \( x - 5 \) 10 \( \frac{4}{(2x + 1)^2} \)
11 \( x + 8 \) 12 \( \frac{x}{2} \)
13 \( \frac{x}{x + 2} \) 14 \( \frac{x + 2}{x + 7} \)
15 \( \frac{x + 2}{x + 4} \) 16 \( \frac{x + 5}{x + 3} \)
17 \( \frac{x + 3}{x + 4} \) 18 \( \frac{x + 4}{x + 2} \)
19 \( \frac{x + 5}{x - 3} \) 20 \( \frac{1}{x - 4} \)

Exercise 23.4 [page 457]
1 \( x = 2 \) and \( y = 2 \) or \( x = -1 \) and \( y = -1 \)
2 \( x = 3 \) and \( y = 10 \) or \( x = -2 \) and \( y = 5 \)
3 \( x = 3 \) and \( y = 19 \) or \( x = -1 \) and \( y = 3 \)
4 \( x = 2 \) and \( y = 20 \) or \( x = \frac{1}{2} \) and \( y = \frac{1}{2} \)
5 \( x = 4 \) and \( y = 17 \) or \( x = 0 \) and \( y = 1 \)
6 \( x = 2 \) and \( y = 0 \) or \( x = -1 \) and \( y = -3 \)
7 \( x = 3 \) and \( y = 1 \) or \( x = -1 \) and \( y = -3 \)
8 \( x = 2 \) and \( y = 2 \) or \( x = -\frac{1}{2} \) and \( y = -2\frac{1}{2} \)
9 \( x = 3 \) and \( y = -1 \) or \( x = 1 \) and \( y = -3 \)
10 \( x = 1 \) and \( y = -6 \) or \( x = 6 \) and \( y = -1 \)
11 \( x = 5 \) and \( y = 13 \) or \( x = -3 \) and \( y = -3 \)
12 \( x = 6 \) and \( y = 1 \) or \( x = -5 \) and \( y = -10 \)
13 \( x = 1 \) and \( y = 2 \) or \( x = 2 \) and \( y = 4 \)
14 \( x = 2 \) and \( y = -3 \) or \( x = 3 \) and \( y = -1 \)
15 \( x = 5 \) and \( y = 3 \) or \( x = 0.6 \) and \( y = -5.8 \)
16 \( x = 1 \) and \( y = 2 \) or \( x = -2 \) and \( y = -1 \)

Exercise 23.5 [page 458]
1 \( x = \frac{5}{3 - m} \) 2 \( x = \frac{d - b}{a - c} \)
3 \( x = \frac{2k}{2 - k} \) 4 \( y = \frac{1 - 2d}{d - 1} \)
5 \( t = \frac{bc - a}{1 - c} \) 6 \( x = \frac{n + 2}{3 - k} \)
7 \( x = \frac{ab}{1 - 5b} \) 8 \( x = \frac{3}{a - 2} \)
9 \( x = \frac{ka}{1 - k} \) 10 \( u = \frac{vf}{v - f} \)

Exercise 23.6 [page 459]
1 Let the numbers be \( n \) and \( n + 1 \).
Their sum is \( 2n + 1 \) which is odd.
2 Let the numbers be \( 2n \) and \( 2m \).
Their product is \( 4mn = 2 \times 2mn \), hence even.
3 Let the numbers be \( 2n + 1 \) and \( 2m + 1 \).
Their product is \((2n + 1)(2m + 1) = 4mn + 2n + 2m + 1 = 2 \times (2mn + n + m) + 1\), hence odd.
4 Let the numbers be \( n, n + 1 \) and \( n + 2 \).  
Their sum is \( n + n + 1 + n + 2 = 3n + 3 \)  
\[ = 3 \times (n+1), \]
hence a multiple of 3.

5 Let the numbers be \( 2n + 1 \) and \( 2m + 1 \).  
Then  
\[ (2n+1)^2 - (2m+1)^2 = [4n^2 + 4n + 1] - [4m^2 + 4m + 1] \]
\[ = 4n^2 + 4n + 1 - 4m^2 - 4m - 1 \]
\[ = 4n^2 + 4n - 4m^2 - 4m \]
\[ = 4(n^2 + n - m^2 - m), \]
hence a multiple of 4.

6 a) \( 4 \times \frac{1}{2}ab = 2ab \)  
b) i) \( c^2 + 2ab \)  
ii) \( a^2 + b^2 + 2ab \)  
d) Pythagoras’ theorem

7 b) Setting \( x = 3 \) gives \( 301 \times 299 = 89 \, 999, \) so not prime.

8 Let the consecutive odd numbers be \( 2n - 1 \) and \( 2n + 1 \).  
Then  
\[ (2n+1)^2 - (2n-1)^2 = [4n^2 + 4n + 1] - [4n^2 - 4n + 1] \]
\[ = 4n^2 + 4n + 1 - 4n^2 + 4n - 1 \]
\[ = 8n, \] hence a multiple of 8.

Review Exercise 23 (page 460)

2 \( a = 3 \)

3 a) 10  
b) 3  
c) 2

4 \( \sqrt{22} \)

5 a) i) 3.5  
ii) 1

b) 3

6 a) 4  
b) 2  
c) 83\frac{1}{3} \%

7 a) \( \frac{3x}{(x-2)(x+4)} \)  
b) 8, -1

8 10\frac{1}{2}

9 a) 5n

b) Two consecutive multiples of 5 are \( 5n \) and \( 5(n + 1) \)

i) \( 5n + 5(n + 1) = 5n + 5n + 5 = 10n + 5; \) 10n is even for all integer values of \( n, \) hence 10n + 5 is odd as 5 is odd and an even number + odd number = odd number.

ii) \( 5n \times 5(n + 1) = 25n(n + 1); \) when \( n \) is odd then \( (n + 1) \) is even and when you multiply by an even number the result is even; likewise, when \( n \) is even the product is even.

10 \( (n+1)^2 - (n-1)^2 = (n^2 + 2n + 1) - (n^2 + 2n - 1) = 4n \)  
\( 4n \) is a multiple of 4 for all values of \( n \)

11 a) \( (2a - 1)^2 - (2b - 1)^2 = (4a^2 - 4a + 1) - (4b^2 - 4b + 1) \)
\[ = 4a^2 - 4b^2 - 4a + 4b \]
\[ = 4(a-b)(a+b) - 4(a-b) \]
\[ = 4(a-b)(a+b - 1) \]

b) The difference between the squares of any two odd numbers is  
\( (2a - 1)^2 - (2b - 1)^2 = 4(a-b)(a+b - 1) \)  
which is a multiple of 4.

To prove \( (2a - 1)^2 - (2b - 1)^2 \) is a multiple of 8 then need to show that \( (a-b)(a+b - 1) \) is a multiple of 2 (since \( 8 = 4 \times 2 \).

When \( a \) and \( b \) are either both even or both odd then \( (a-b) \) is even and so \( (a-b) \) is a multiple of 2.

When only one of \( a \) or \( b \) is even then \( (a+b - 1) \) is even and so \( (a+b - 1) \) is a multiple of 2.

Hence \( 4(a-b)(a+b - 1) \) is a multiple of 8 as required.

12 2 is prime and \( 2^2 + 3 = 7 \).

13 a) \( (x+1)(2x+5) \)  
b) \( \frac{11x+15}{(x+1)(2x+5)} \)

14 a) \( 23 - 6x \)  
b) \( 32x^2y^{15} \)  
c) \( \frac{2(n-1)}{n-2} \)

15 a) 7  
b) \( \frac{2x}{2x+3} \)

16 \( y = \frac{2k}{4 + 3k} \)

17 \( x = \frac{ay}{y+1} \)

18 \( x = 2 \) and \( y = 5 \) or \( x = -1.4 \) and \( y = -5.2 \)

19 a) If \( y = 6 \) then \( x^2 = -11 \) so Bill must be wrong.  
b) \( x = 3 \) and \( y = 4 \) or \( x = -1.4 \) and \( y = -4.8 \)

Internet Challenge 23 (page 463)

1 Pythagoras’ theorem

2 Circumference of a circle

3 Area of a trapezium

4 Voltage = Current \( \times \) Resistance

5 Volume of a cone

6 Quadratic equation formula

7 Energy = mass \( \times \) (speed of light)\(^2\)

8 Surface area of a sphere

9 Distance \( s \) in terms of initial speed \( u, \) acceleration \( a \) and time \( t \)

10 Periodic time for a pendulum of length \( l \)

11 Euler’s formula for faces, edges and vertices of a polyhedron

12 Conversion from degrees Fahrenheit to degrees Celsius

13 Kinetic energy

14 Potential energy

15 Optics formula, \( u = \) object distance,  
\( v = \) image distance, \( f = \) focal length

16 Electrical resistance (resistors in parallel)

17 Simple interest

18 Area of a triangle

19 Gravitational force of attraction

20 Work done by a force \( F \) moving over a distance \( d \)